

**PROBABILITY-OF-DAMAGE PROBLEMS OF FREQUENT OCCURRENCE,**

**CENTER FOR NAVAL ANALYSES ALEXANDRIA VA**

**11 DEC 1959**

**Distribution Statement A:  
Approved for public release. Distribution is unlimited.**



DEPARTMENT OF THE NAVY  
OFFICE OF THE CHIEF OF NAVAL OPERATIONS  
WASHINGTON 25, D. C.

IN REPLY REFER TO  
Cp03EG/sjb  
Ser 70P03

11 DEC 1959

From: Chief of Naval Operations  
To: Distribution List

Subj: Operations Evaluation Group Study No. 626; forwarding of

Encl: (1) OEG Study No. 626, Unclassified, entitled "Probability-  
of-Damage Problems of Frequent Occurrence"

1. OEG Study No. 626 (enclosure (1)), prepared by the Operations Evaluation Group, is forwarded herewith for retention.
2. This study summarizes some of the salient methodology which has been developed for the solution of problems involving the computation of the probability of damage to military targets.
3. The Chief of Naval Operations recommends that this study be reviewed by those responsible for analyses which include the derivation of the probability of damage.
4. If additional copies are required, they will be forwarded upon request.

DISTRIBUTION:  
Attached list

*R. Mason*  
R. MASON  
By direction

DISTRIBUTION LIST FOR OPNAV SERIAL 70P03:

<u>CNO</u>	<u>SNDL</u>
<u>Op92</u> (6)	<u>21</u> (CINCPACFLT, LANTFLT, NELM)
93	22 (FLTCDRS)
51	24A (NAVAIRPAC, LANT)
33	26F (OPTEVFORCOMDS (including KWTED,
34	NIDEVDET: MinWar-
50	EvalDet; GMEU-1))
52	
<u>O6C</u>	<u>OTHER</u>
<u>07T</u>	<u>DIRWSEG</u> (2)
	SENNAVMBRWSEG
<u>BUSHIPS</u>	DODR&E
<u>100</u> (5)	COMASDEFORLANT
	PRESNAVWARCOL (5)
<u>Chief, BUWEPs</u> (10)	SUPTNAVPGSCOL (5)
	DIRNRL
<u>BUAER</u>	CO&DIRNRL
<u>AER</u> (10)	COMNOLS
RS	CO&DIR DTMB
	COMNOTS (2)
<u>BUORD</u>	COMNPG
<u>A</u> (10)	CNC
Re	INSORD, APL/JHU
Req	Dept of AF (Attn: AFCAS) (7)
Ree	Dept of Army (Attn: Adj Gen) (6)
	DIR, Long Range Studies Project, NWC
<u>ONR</u>	The RAND Corporation (via USAF Liaison
<u>100</u>	Office) (3)
400	
430	<u>ORO/JHU</u> (3)
432	
436	
491	
492	
493	

UNCLASSIFIED

OPERATIONS EVALUATION GROUP

STUDY NO. 626

PROBABILITY-OF-DAMAGE PROBLEMS OF FREQUENT OCCURRENCE (U)

11 December 1959

This Study represents the view of the Operations Evaluation Group at the time of issue. It does not necessarily reflect the official opinion of the Chief of Naval Operations except to the extent indicated in the forwarding letter. It includes information of an operational rather than a technical nature, and should be made available only to those authorized to receive such information.

Enclosure 1  
CNO Ser 70P03  
Dated 11 December 1959

Prepared by the  
OPERATIONS EVALUATION GROUP  
Office of the Chief of Naval Operations

UNCLASSIFIED

## ABSTRACT AND INTRODUCTION

Problems involving computation of the probability of damage, or the expected fraction of the target damaged, occur frequently in military operations research. These problems have been worked on by many people over a long period of time. This study is intended to summarize some of the more important results and to indicate sources of information on other problems. No attempt has been made to include an extensive bibliography or to track down the first man who derived any particular result.

Although many of the results were known long ago - some of them during the 19th century or earlier - it was not until World War II that systematic efforts were made to obtain answers for a wide variety of situations. Recently several reviews have been made of this material. One of these is reference (a), which does not include some of the results given here. On the other hand, reference (a) gives results on the inverse problem, that of determining the probable location of the aiming point from the locations of the burst points or from the damage produced. The inverse problem is of less interest than the direct problem; the results are of little usefulness in most damage problems.

### SINGLE-SHOT PROBABILITIES

In this section we will consider the problem of computing the probability of killing the target, or of inflicting a stated degree of damage, by a single weapon (projectile, bomb, etc.). Because of its frequent occurrence the two-dimensional case will be used throughout for the illustration of methods and statements of results. In most cases the manner in which the results would be adapted to one dimension or to more than two dimensions should be evident. The derivation must be examined, however, in every instance in which only the results are given.

We will consider both point and area targets, a point target being one whose dimensions are small compared with the "damage radius" of the weapon relative to the target. We will also consider two types of "conditional damage function", that is, the function that describes the probability of damaging the target if the weapon detonates at a given point relative to the target. These two functions are usually referred to as the cookie-cutter function and the Gaussian function. In the former case the probability of damage is 1 within some area about the point of detonation and is zero outside this area. In the latter the probability of damage varies gradually from a value of 1 at the point of detonation, decreasing toward zero as the distance from the point of detonation increases.

These are some of the principal results:

#### ★ SINGLE POINT TARGET

Consider the target at the origin of a rectangular coordinate system in a suitable plane. This plane might be the horizontal plane or a plane normal to the trajectory of a weapon which would pass through the target. Let the conditional probability of damage to the target, if the weapon is fired or released on a trajectory through the point  $(x, y)$ , be  $p_d(x, y)$ . Let the joint probability density function of  $x$  and  $y$  be  $f(x, y)$ . The joint density function  $f(x, y)$  describes the density of shots or bomb-drops in the target plane. It is non-negative and is so chosen that the

integral of it over any area is the probability that the weapon will fall in this area. Then the probability of damage to the target from a single weapon chosen at random is

$$P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_d(x, y) f(x, y) dx dy .$$

Some cases are given below:

● Circular Cookie-Cutter Damage Function:

$$p_d(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases} .$$

For this function the probability of damage is

$$P = \int \int_{\text{circle}} f(x, y) dx dy .$$

where the integral is to be taken over the circle of radius  $R$  with center at the origin.

■ Probability of damaging single point target with a single shot when the conditional damage function is the circular cookie-cutter function and when the distribution of shots is circular normal and centered on the target:

More precisely, we assume that  $x$  and  $y$  are independent variables having normal distributions with means zero and common standard derivation,  $\sigma$  . Then  $f(x, y)$  has the form

$$f(x, y) = \frac{1}{2 \pi \sigma^2} \exp \left[ - (x^2 + y^2) / 2 \sigma^2 \right] .$$

By elementary integration we find that

$$P = 1 - \exp(-R^2 / 2 \sigma^2) .$$

■ Probability of damaging single point target with a single shot when the conditional damage function is the circular cookie-cutter function and when the distribution of shots is elliptical normal and centered on the target:

This case is the same as the one preceding, except that the two standard deviations are unequal. This slight change makes the integration difficult. If we change to polar coordinates  $(r, \theta)$  and integrate with respect to  $\theta$  we obtain the following result:

$$\begin{aligned}
P &= \frac{1}{2\pi\sigma_x\sigma_y} \int_0^{2\pi} \int_0^R \exp \left[ -r^2 \cos^2 \theta / 2 \sigma_x^2 - r^2 \sin^2 \theta / 2 \sigma_y^2 \right] r dr d\theta \\
&= \frac{1}{2\pi\sigma_x\sigma_y} \int_0^R e^{-ar^2} r dr \int_0^{2\pi} e^{-b r^2} \cos \phi d\phi, (\phi = 2\theta) \\
&= \frac{1}{\sigma_1\sigma_2} \int_0^R e^{-ar^2} I_0(br^2) r dr \\
&= \frac{1}{2\sigma_1\sigma_2} \int_0^{R^2} e^{-au} I_0(bu) du
\end{aligned}$$

Here  $\sigma_1$  is the smaller of  $\sigma_x, \sigma_y$  and  $\sigma_2$  is the larger of these. Also,

$$a = \frac{1}{4} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right), \quad b = \frac{1}{4} \left( \frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} \right)$$

and  $I_0(x)$  is the modified Bessel function of zero order. If we replace the Bessel function by its series expansion and integrate term by term, we get the following result:

$$P = \frac{\sigma_1}{\sigma_2} \sum_{k=0}^{\infty} (-1)^k \frac{(R^2/2\sigma_1^2)^{k+1}}{(k+1)!} \sum_{i=0}^k (-1/4)^i \binom{k}{i} \binom{2i}{i} (1 - \sigma_1^2/\sigma_2^2)^i.$$

For cases in which  $R$  is small compared with  $\sigma_1$  this series converges rapidly.

The integral of the elliptical normal distribution over a circle, as derived above, is equal to the integral of the circular normal distribution with unit standard deviation over the ellipse that has semi-axes  $R/\sigma_x$  and  $R/\sigma_y$ .

The probability of damage in the case of unequal standard deviations can be approximated by

$$P = 1 - \exp(-R^2/2\sigma^2)$$

where  $\sigma$  is an appropriate function of  $\sigma_1$  and  $\sigma_2$ . Functions frequently used are the geometric mean  $\sigma_g$ , the arithmetic mean  $\sigma_a$ , and the root-mean-square  $\sigma_s$ .

These are defined as follows:

$$\sigma_g^2 = \sigma_1 \sigma_2$$

$$\sigma_a = (\sigma_1 + \sigma_2)/2$$

$$\sigma_s^2 = (\sigma_1^2 + \sigma_2^2)/2$$

It is evident that  $\sigma_a$  lies between  $\sigma_g$  and  $\sigma_s$ ; in fact,  $\sigma_a^2$  is the mean of  $\sigma_g^2$  and  $\sigma_s^2$ . Let  $P_g$ ,  $P_a$ , and  $P_s$  be the probabilities of damage given by

$P = 1 - \exp(-R^2/2\sigma^2)$  when  $\sigma$  has the values  $\sigma_g$ ,  $\sigma_a$ , and  $\sigma_s$  respectively. Of these 3 approximations the best seems to be:

$$P_g \text{ when } 0 \leq R^2 < 0.5 \sigma_a^2$$

$$P_a \text{ when } 0.5 \sigma_a^2 \leq R^2 \leq 2.6 \sigma_a^2$$

$$P_s \text{ when } 2.6 \sigma_a^2 < R^2$$

More accurate approximations can be obtained by using linear combinations of these probabilities. For example, use  $P = (P_g + P_a)/2$  in the interval (0, 0.9) for  $R^2/\sigma_a^2$ ,  $P = P_a$  in the interval (0.9, 1.9),  $P = (P_a + P_s)/2$  in the interval (1.9, 3.4), and  $P = P_s$  in the interval (3.4,  $\infty$ ). This approximation is accurate to about  $\sigma_2/100 \sigma_1$  for  $0.2 \leq \sigma_1/\sigma_2 \leq 1.0$ .

For smaller values of  $\sigma_1/\sigma_2$  the approximation obtained by assuming  $\sigma_1 = 0$  is more accurate; that is,  $P$  is approximated by the one-dimensional radial distribution of standard deviation  $\sigma_2$ :

$$\begin{aligned} P &\approx \left(1/\sigma_2 \sqrt{2\pi}\right) \int_{-R}^R \exp(-x^2/2\sigma_2^2) dx \\ &= \left(2/\sqrt{2\pi}\right) \int_0^{R/\sigma_2} \exp(-t^2/2) dt \\ &= 2 \Phi(R/\sigma_2) - 1 \end{aligned}$$



where  $\Phi$  is the normal error integral:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$

- Probability of damaging single point target with a single shot when the conditional damage function is the circular cookie-cutter function and when the distribution of shots is elliptical normal and  $x$  and  $y$  are correlated:

We assume that the means are zero but  $x$  and  $y$  are not independent. Let the covariance be  $\sigma_{xy}$ . This is equal to the expected value\* of  $xy$  and is equal to  $\rho_{xy} \sigma_x \sigma_y$  where  $\rho_{xy}$  is the correlation coefficient. The probability of damage in this case can be obtained from the above results by replacing  $\sigma_1^2$  and  $\sigma_2^2$  by  $\lambda_1$  and  $\lambda_2$  respectively, where  $\lambda_1$  and  $\lambda_2$  are the roots of the equation:

$$(\sigma_x^2 - \lambda)(\sigma_y^2 - \lambda) - \sigma_{xy}^2 = 0$$

with  $\lambda_1 \leq \lambda_2$ . The roots of this equation are often called the eigen-values of the covariance matrix. The equation is obtained by equating to zero the determinant in which the elements are the corresponding elements of the covariance matrix with the elements of the main diagonal reduced by  $\lambda$ .

This result can be obtained easily from the joint density function, which is:

$$f(x, y) = (1/2\pi \sigma_x \sigma_y \sqrt{1-p^2}) \cdot \exp \left[ -(\sigma_y^2 x^2 - 2p \sigma_x \sigma_y xy + \sigma_x^2 y^2) / 2 \sigma_x^2 \sigma_y^2 (1-p^2) \right]$$

where  $\sigma_{xy} = p \sigma_x \sigma_y$ . We now rotate the axes through one of the angles that eliminates the cross-product term in the quadratic form. The transformation is linear with determinant equal to 1. If  $u$  and  $v$  are the new rectangular coordinates, the function  $f(x, y)$  reduces to

$$(1/2\pi \sqrt{\lambda_1 \lambda_2}) \exp (-u^2/2 \lambda_1 - v^2/2 \lambda_2)$$

where  $\lambda_1$  and  $\lambda_2$  are the roots of the equation given above. Since the transformation does not alter the conditional damage function, the results can be obtained from the previous case by replacing  $\sigma_1^2$  and  $\sigma_2^2$  by  $\lambda_1$  and  $\lambda_2$ .

\* Because the means are zero. More generally, the covariance is equal to the expected value of  $(x - \bar{x})(y - \bar{y})$ , where  $\bar{x}$  and  $\bar{y}$  are the means of  $x$  and  $y$  respectively.

- Probability of damaging single point target with a single shot when the conditional damage function is the circular caustic-center function and when the distribution of shots is circular normal and is centered at a point  $k$  units from the target

Here we assume again that  $x$  and  $y$  are independent and are normally distributed with equal standard deviations, but the means of  $x$  and  $y$  are not both zero. Because of the circular conditional damage function, we need be concerned only with the distance  $k$  of the center of the shot distribution from the target. The integral in this case is approximately as difficult as that of the elliptical distribution centered at the target. If we transform to polar coordinates and integrate with respect to  $\theta$  we obtain the following result:

$$P = \frac{1}{\sigma^2} \int_0^R e^{-(x^2 + y^2)/2\sigma^2} \frac{1}{2\sigma^2} e^{-k^2/2\sigma^2} r dr d\theta \\ = e^{-k^2/2\sigma^2} \int_0^R e^{-r^2/2\sigma^2} \int_0^{2\pi} e^{-k^2/2\sigma^2} r dr d\theta$$

Again we can expand the Bessel function into a series and integrate term by term. This can be written in a number of forms of which two are given below:

$$P = e^{-k^2/2\sigma^2} \sum_{n=0}^{\infty} \frac{(k^2/2\sigma^2)^n}{n!} \int_0^R e^{-r^2/2\sigma^2} r dr d\theta \\ = e^{-k^2/2\sigma^2} \sum_{n=0}^{\infty} \frac{(k^2/2\sigma^2)^n}{n!} \int_0^R e^{-r^2/2\sigma^2} r dr d\theta$$

Here,  $\Gamma$  is the incomplete gamma function tabulated in reference (13). It is evident that  $P$  can be written as a function  $P(R, \sigma, k, \sigma)$  of the two ratios listed. Tables of this function are given in reference (13) and at numerous other places. An approximation that is useful for small values of  $R$  is the following:

$$P \approx \frac{R^2}{R^2 + 4\sigma^2} e^{-k^2/2\sigma^2} \quad R < \sigma$$

This approximation is in error by less than .007, and usually by less than .001, for values of  $R$  less than  $\sigma$ . For values of  $R$  less than  $1.2\sigma$  the error is less than .02 and usually less than .01. For very large values of  $R$ , greater than  $5\sigma$ , a good approximation is the following:

2 - 10

3. Probabilities of destroying single point targets with a single shot when the distribution density function is the Gaussian random number function and when the distribution of shots is a Poisson random variable centered in the target.

- Circular Gaussain Surface Function.

This function was first introduced, as far as is known, by van der Waerden while working as a consultant for the Ballistics Research Laboratory. In some cases it is a more realistic description of the conditional damage function than is the simple-curve function. However, its biggest advantage is the ease with which the integrals can be obtained and the simplicity of the resulting formulas. The formula is sometimes written with a factor 2 in the denominator of the exponent. This is done to simplify some of the formulas slightly. It appears preferable that the formula be written in the form given above, since in this form the quantity  $a$  is the equivalent damage radius, in the sense that the integral of the function over the entire plane is the area of a circle of radius  $a$ . That is,

$$P_1(1) = 1$$

In this case the damage function and the density function depend only on  $x$  rather than on  $\mathbf{x}$ . When we change to polar coordinates, we find the probability density

$$= \frac{1}{2} (2^2 - 2 \times 2^2) = -1$$

- Probability of damaging single point target with a single shot when the conditional damage function is the circular Gaussian function and when the distribution of shots is elliptical normal and centered on the target:

Again the integral is found in a straightforward manner by separating it into a product of 2 integrals by the separation of variables. The probability is

$$P = a^2 / \sqrt{(a^2 + 2\sigma_x^2)(a^2 + 2\sigma_y^2)} .$$

- Probability of damaging single point target with a single shot when the conditional damage function is the circular Gaussian function and when the distribution of shots is circular normal and is centered at a point h units from the target:

Again the integral can be computed readily by the separation of variables. The probability is given by

$$P = \left[ a^2 / (a^2 + 2\sigma^2) \right] \exp \left[ -h^2 / (a^2 + 2\sigma^2) \right] .$$

Exactly the same expression is obtained for the case in which the distribution is the Cauchy distribution with density function

$$f(x, y) = (1/2\pi\sigma^2) \exp \left[ -(x^2 + y^2 + h^2)/2\sigma^2 \right] I_0 \left( h \sqrt{x^2 + y^2} / \sigma^2 \right) .$$

- Probability of damaging single point target with a single shot when the conditional damage function is the circular Gaussian function and when the distribution of shots is elliptical normal not centered on the target:

Assume that the center of the distribution is at  $(h_x, h_y)$ . Again the integral can be separated, and the probability of damage is

$$P = \left[ a^2 / \sqrt{(a^2 + 2\sigma_x^2)(a^2 + 2\sigma_y^2)} \right] \cdot \exp \left[ -h_x^2 / (a^2 + 2\sigma_x^2) - h_y^2 / (a^2 + 2\sigma_y^2) \right] .$$

- Elliptical Gaussian Damage Function:

$$p_d(x, y) = \exp \left( -x^2/a_x^2 - y^2/a_y^2 \right) .$$

In this case the integral of the conditional damage function over the entire plane is  $\pi a_x a_y$ , the area of an ellipse with semi-axes  $a_x$  and  $a_y$ . We can determine the probabilities in this case from the corresponding probabilities for the circular Gaussian damage function by making the following substitutions:

Replace  $a^2$  in the numerator by  $a_x a_y$

Replace in the denominators:

$$a^2 + 2\sigma_x^2 \text{ by } a_x^2 + 2\sigma_x^2$$

$$a^2 + 2\sigma_y^2 \text{ by } a_y^2 + 2\sigma_y^2$$

$$a^2 + 2\sigma^2 \text{ by } \sqrt{(a_x^2 + 2\sigma^2)(a_y^2 + 2\sigma^2)}$$

#### ● Other Conditional Damage Functions:

For some weapons and weapons effects the conditional damage function appears to lie somewhere between the circular cookie-cutter function and the circular Gaussian function. In appendix A, a sequence of functions of this type is proposed. The first member of this sequence is the Gaussian function, and the limiting member is the cookie-cutter function. These functions have the property that the probability integrals for the usual cases can be computed in finite form in terms of known functions. The second member of the sequence yields probabilities that are approximately halfway between the corresponding probabilities for the Gaussian and cookie-cutter functions. This property could be used as a means of approximating the difficult cases for the cookie-cutter function. However, since the second member of the sequence probably is closer to the true conditional damage function than is the cookie-cutter function, it is more reasonable to use the probabilities given by this second member of the sequence.

#### ★ SEVERAL POINT TARGETS

When several point targets exist, there is generally a need to answer questions of the following type: What is the probability of damaging all the targets, or at least a specified number of them, with a single weapon? (The case of several weapons will be discussed later.) To answer, we must first compute the desired probability under the condition that the weapon detonates at a particular point  $(x, y)$ . This quantity is then treated as the conditional probability in the previous work and we must average this function over the shot distribution.

For the cookie-cutter damage function the integrals involved are invariably difficult. The integrals are fairly easy to compute, however, in the case of the Gaussian damage function. We give one example below.

Assume that the distribution of shots is circular normal of standard deviation  $\sigma$  and is centered at a point midway between two targets. Assume that the targets are located at the points  $(-h, 0)$  and  $(h, 0)$ . If the weapon detonates at the point  $(x, y)$  the probability of damaging both targets is

$$\begin{aligned} P_d(x, y) &= \exp \left\{ - \left[ (x - h)^2 + y^2 + (x + h)^2 + y^2 \right] / a^2 \right\} \\ &= \exp \left[ -2(x^2 + y^2 + h^2)/a^2 \right] \end{aligned}$$

To obtain the desired probability we must now multiply by the density function and integrate over the entire plane. The result:

$$P(\text{both}) = \left[ a^2 / (a^2 + 4\sigma^2) \right] \exp(-2h^2/a^2) .$$

#### ★ AREA TARGET

In this case we assume that the target is spread over a large area, the maximum dimension of this area being at least as large as the damage radius of the weapon. It is assumed that, for a large fraction of the detonations, some parts of the target would be damaged critically while other parts would be left undamaged. In this case it makes little sense to talk about probability of damage. Instead we must consider the problem of determining the expected fraction of the target that will be damaged, or the probability of damaging at least a given fraction of the target.

Assume that the burst occurs at the point  $(x, y)$ . Then if  $(X, Y)$  is any point of the target, the conditional probability of inflicting damage to this point is

$$p_d(x - X, y - Y) .$$

Then the conditional fraction  $p(x, y)$  damaged if the burst occurs at  $(x, y)$  is

$$p(x, y) = (1/A) \iint p_d(x - X, y - Y) dX dY$$

where the integral is taken over the target of area  $A$ . The expected fraction of the target damaged is

$$F = \iint_{-\infty}^{\infty} p(x, y) f(x, y) dx dy$$

where, as before,  $f(x, y)$  is the joint density function.

Let  $S_0$  be the set of points

$$S_0: \text{Points } (x, y) \text{ such that } p(x, y) \geq F_0 .$$

Then the probability of damaging at least the fraction  $F_0$  is equal to

$$P(\geq F_0) = \iint_{S_0} f(x, y) dx dy$$

where the integral is taken over the set  $S_0$ . Sandia Corporation literature describes an analogue computer which was designed to obtain this function for a wide variety of target complexes and density functions.

We illustrate these ideas with one example. Let the target be a circle of radius  $T$  and let the conditional damage function be the circular Gaussian function of equivalent radius  $a$ . Assume that the distribution is circular normal of standard deviation  $\sigma$  and centered at the center of the target. Then the conditional fraction of the target damaged is

$$p(x, y) = (1/\pi T^2) \int_0^{2\pi} \int_0^T \exp \left\{ - \left[ (r \cos \theta - x)^2 + (r \sin \theta - y)^2 \right] / a^2 \right\} r dr d\theta$$

$$= (a^2/T^2) \exp \left[ - (x^2 + y^2)/a^2 \right] \int_0^{T/a^2} e^{-u} I_0(2 \sqrt{u(x^2 + y^2)/a}) du.$$

The expected fraction of the target damaged is

$$F = (1/2 \pi \sigma^2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \exp \left[ - (x^2 + y^2)/2 \sigma^2 \right] dx dy$$

$$= (a^2/T^2) \left\{ 1 - \exp \left[ - T^2/(a^2 + 2 \sigma^2) \right] \right\}.$$

To obtain the probability of damaging at least the fraction  $F_0$  we would have to solve the equation obtained by equating  $p(x, y)$  to  $F_0$ . This will be a circle whose radius could be obtained by trial and error. Let the radius of this circle be  $R_0$ . Then the probability of damaging at least this fraction would be equal to

$$P(\geq F_0) = \begin{cases} 1 - \exp(-R_0^2/2\sigma^2), & 0 \leq F_0 \leq \frac{a^2}{T^2} \left[ 1 - \exp(-T^2/a^2) \right] \\ 0, & F_0 > \frac{a^2}{T^2} \left[ 1 - \exp(-T^2/a^2) \right]. \end{cases}$$

If we replace the circle target by an equivalent Gaussian target of density

$$\exp \left[ - (X^2 + Y^2)/T^2 \right]$$

at the point  $(X, Y)$ , the formulas become simpler. Thus we find

$$p(x, y) = \left[ a^2/(a^2 + T^2) \right] \exp \left[ - (x^2 + y^2)/(a^2 + T^2) \right]$$

$$F = a^2/(a^2 + T^2 + 2 \sigma^2)$$

$$P(\geq F_0) = \begin{cases} 1 - \left[ F_0 (a^2 + T^2)/a^2 \right]^{(a^2 + T^2)/2\sigma^2} & , 0 \leq F_0 \leq a^2/(a^2 + T^2) \\ 0 & , F_0 > a^2/(a^2 + T^2) \end{cases}$$

### SALVOS

So far we have considered the probability of damage with a single weapon only. If several weapons are dropped separately in such a way that the probabilities are independent, we can combine the single-shot probabilities by well known methods. An example of partial correlation is that of a salvo of weapons, that is, a group of weapons released simultaneously or nearly simultaneously. In this case the weapons in the salvo have a particular distribution which is not the same in all respects as the distribution that would be obtained on another salvo. A characteristic of the distribution that might vary from salvo to salvo is the point about which the individual weapons in the salvo are grouped, such as the mean point of impact (MPI). The weapons of a particular salvo will have a particular MPI but this MPI may vary from salvo to salvo. In focusing attention on the MPI we do not intend to imply that this is the only characteristic of the distribution of weapons in a salvo that might vary from salvo to salvo. In the formulation below, however, we shall consider only the MPI.

Let  $f(x, y | u, v)$  be the joint density function of the point  $(x, y)$  of detection of a weapon in a salvo that has its MPI at the point  $(u, v)$ . By the methods discussed previously we can find the single-shot probability, given that the MPI is  $(u, v)$ . Let this be  $P = P(u, v)$  and let  $g(u, v)$  be the joint density function of  $u$  and  $v$ . Let  $F[P]$  be any functional of  $P$  of interest. Then the average value of this quantity is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F[P(u, v)] g(u, v) du dv$$

For instance we might want to find the probability of scoring at least one damaging hit with a salvo. If the probabilities of damage by weapons in a salvo are independent, except for the common MPI, and  $n$  weapons are used in a salvo, then the quantity of interest is

$$F[P] = 1 - (1 - P)^n = \sum_{i=1}^n (-1)^{i-1} \binom{n}{i} P^i(u, v)$$

From this we can readily find the probability of scoring at least one damaging hit by integrating the product of this function of  $u$  and  $v$  and  $g(u, v)$  over the  $(u, v)$  plane.



We will treat in full one particular example, perhaps the simplest example possible. Assume that the conditional damage function is circular Gaussian with equivalent radius  $a$ . Assume that the distribution of shots in a salvo is circular normal of standard deviation  $s$  and is centered at  $(u, v)$ . Also, assume that the distribution of the MPI is circular normal with standard deviation  $\sigma$  and is centered at the target. Then from the solution to the problem concerning the circular normal distribution of shots centered at a point  $h$  units from the target, we have

$$P(u, v) = A \exp \left[ - (u^2 + v^2) A / a^2 \right], \quad A = a^2 / (a^2 + 2s^2) \quad .$$

Also

$$\iint_{-\infty}^{\infty} P^i(u, v) g(u, v) du dv = A^i / (1 + i/B), \quad B = (a^2 + 2s^2) / 2\sigma^2 \quad .$$

The latter result is obtained by straightforward integration. Finally, the probability of scoring at least one damaging hit can be written as

$$P(\geq 1) = \sum_{i=1}^{\infty} (-1)^{i-1} \binom{n}{i} A^i / (1 + i/B)$$

where

$$A = a^2 / (a^2 + 2s^2), \quad B = (a^2 + 2s^2) / 2\sigma^2 \quad .$$

The probability of at least 1 damaging hit in this case is approximated by assuming that  $P$  is so small that we can estimate the probability of at least 1 damaging hit when the MPI is  $(u, v)$  by

$$F(P) \approx 1 - \exp(-nP), \quad P = P(u, v) \quad .$$

Using this approximation we can express the average value in the form

$$P(\geq 1) = 1 - (1/\sigma^2) \int_0^{\infty} \exp \left[ -r^2/2\sigma^2 - nA \exp(-r^2 A/a^2) \right] r dr \quad .$$

Letting  $t = nA \exp(-r^2 A/a^2)$  we obtain

$$P(\geq 1) \approx 1 - (nA)^{-B} \int_0^{nA} e^{-t} t^{B-1} dt = 1 - (nA)^{-B} \left[ (B+1) \Gamma \left( (nA)^{-B} / (B+1), B+1 \right) \right]$$

where  $\Gamma(B+1)$  is the complete gamma function and  $\Gamma(u, p)$  is the incomplete gamma function tabulated in reference (b).

The above example indicates the general procedure in problems of this kind. The reader may wish to compare the probability obtained in the above example with that obtained by first finding the over-all distribution of weapons, then determining the probability of damage with a single weapon, and from this deriving the probability of at least one damaging hit with a weapons which are considered independent. The probability of scoring at least one damaging hit by the latter method should be greater than the probability found by the former (and correct) method. Weapons fired in salvos constitute one of the simplest examples of correlation.

### SEQUENCE OF SHOTS

In many problems two or more weapons are fired on a single run or event in such a way that the individual probabilities are not independent and there are many characteristics of the distribution that change from shot to shot. Examples are: surface-to-air anti-aircraft fire by guns, rockets and guided missiles; air-to-air engagements with guns, rockets, and missiles; stock bombing and intervalometer bombing by aircraft, etc. In fact, most of the realistic situations involving more than one weapon are of this type. Usually the correlation effects, both auto-correlation and Leman, are too large to be ignored. Numerous efforts have been made to solve problems of this sort. Most of the solutions require that the process be stationary. But the process is usually not stationary in the problems encountered.

A sequence of shots that forms a simple Markoff chain probably is the simplest type. With a sufficiently long chain this method is adequate to treat the auto-correlation effects. The problem of the Markoff chain, including the limiting law of small numbers (the analogue of the Poisson distribution), is treated in references (c) and (f). There seems to be no easy method of treating Leman effects, except by taking the weighted average of the probabilities for the separate populations.

Submitted by:

*J. M. Dodge*

J. M. DODGE

Operations Evaluation Group

Approved by:

*L. Sternhardt*

L. STERNHARDT

Director

Operations Evaluation Group

- Ref: (a) Andre G. Laurent, Bombing Problems - A Statistical Approach, Operations Research 5, 75 (1957)
- (b) K. Pearson, Tables of the Incomplete Gamma Function, Biometrika Office, University College, London 1934
- (c) RAND RM-330 Uncl 26 Jan 1950
- (d) RAND P-94 Uncl 28 Jul 1949
- (e) B.O. Koopman, A generalization of Poisson's distribution for Markoff chains, Proc. Nat. Acad. Sci., U.S.A. Vol. 36 (1950) pages 202-207
- (f) B.O. Koopman, A law of small numbers in Markoff chains, Trans. Am. Math. Soc. Vol. 70 no. 2 (1951) pages 277-290

## APPENDIX A

### A SEQUENCE OF CONDITIONAL-DAMAGE FUNCTIONS

The conditional damage function is a function which gives the probability that a target which is located at a particular point and bearing from the burst point will suffer at least the stated degree of damage. This probability is a function of distance  $r$  of the center of the target from ground zero, height (depth) of burst, bearing of ground zero relative to the target heading, damage category, target vulnerability, and weapon yield.

It is evident that a function which accurately describes the effects of all these variables will be complicated. First, let the last three variables -- damage category, target vulnerability, and weapon yield -- have assigned values. Then the conditional damage probability is a function of the three coordinate variables -- horizontal distance  $r$ , height (depth), and bearing. First average over the distribution of these variables for a selected aiming point (sometimes called the intended ground zero, IGZ) and intended height of burst.

The most important coordinate variable is horizontal distance  $r$ . For simplicity the conditional probability will be written explicitly as a function of this variable only. It is understood that a particular height, based on weapon yield, effect desired, dud probabilities, etc., has been set in the fuze. It is assumed that errors in height produce a negligible effect on the probability of damage, or that an average over the corresponding error distribution will be taken. Such an average can be taken at this point if the height error is independent of errors in the horizontal plane, which is true for most delivery and fuzing systems in use. However, for air-burst fuzes it is found to be convenient to defer this average until later.

It is assumed that changes in bearing produce negligible changes in damage probability. For some targets, notably ships and aircraft, this is not true. However, in taking the average over the distribution of weapons, these changes with bearing can usually be ignored with negligible error.

When the conditional damage probability is written as a function of  $r$ , it will be assumed that the effects of changes in damage category, target vulnerability, and weapon yield can be described adequately by changes in parameters in this function, without changing the form of the function.

Let  $p(r)$  be the average probability of damage at horizontal distance  $r$  for a given damage category, target type, and weapon yield. Several functions which have been used are the following:

(a) Definite-range or cookie-cutter function:

$$p_0(r) = \begin{cases} 1 & \text{if } r \leq R \\ 0 & \text{if } r > R \end{cases} \quad (\text{A-1})$$

where  $R$ , the damage radius, is a parameter which depends upon damage category, target type, and weapon yield.

(b) Gaussian function:

$$p_1(r) = \exp(-r^2/a_1^2)$$

where " $a_1$ " also is a parameter which depends on the damage variables. A factor of 2 is sometimes used in the denominator of the exponent to simplify later formulas; this should be kept in mind when comparing values of parameters.

The definite-range function appears to be reasonable if all the variables except  $r$  have fixed values. This requires that the damage category be defined precisely, a particular target of known vulnerability be considered, the weapon yield be known exactly, and the height and bearing of the burst position be fixed. As explained above,  $p(r)$  should be an average over some of these variables -- particularly variations in damage which would be included in a given damage category, variations in target vulnerability among targets of a given type, and variations in height of burst unless the average over the height distribution is to be taken later.

For any conditional damage function  $p(r)$ , let  $g(R)$  be the corresponding density function of the damage radius  $R$ . Write  $p_0(r)$  in the form  $p_0(r, R)$  to display the role of  $R$ . Then

$$\begin{aligned} p(r) &= \int_0^\infty p_0(r, R) g(R) dR = \int_r^\infty g(R) dR \\ &= \text{Prob} \left\{ R \geq r \right\} \end{aligned} \quad (\text{A-2})$$

is the distribution function of the damage radius. Hence, the problem of finding the probability function  $p(r)$  is equivalent to that of finding the distribution function of the damage radius  $R$ .

For  $p(r)$  equal to the Gaussian function  $p_1(r)$  the corresponding density function of  $R$  is

$$g_1(R) = -p_1'(R) = \frac{2R}{a_1^2} \exp(-R^2/a_1^2)$$

An objection to this density function is that it gives too much weight to very small and very large values of  $R$ . As will be shown later, only 65 percent of the distribution lies between  $0.5\bar{R}$  and  $1.5\bar{R}$ , where  $\bar{R}$  is the average damage radius.

To get a higher concentration about the average we can multiply the exponential factor by a power of  $R$  greater than one. For example, if

$$g_2(R) = \frac{8R^3}{a_2^2} \exp(-2R^2/a_2^2),$$

the corresponding probability function is

$$p_2(r) = \left(1 + \frac{r^2}{a_2^2}\right) \exp(-r^2/a_2^2)$$

This function and the Gaussian function may be considered as the first two functions of a sequence obtained from the density function

$$p_n(r) = \frac{1}{(n-1)!} \left(\frac{r^2}{a_n^2}\right)^{n-1} \exp(-r^2/a_n^2)$$

The corresponding probability function is

$$p_n(r) = 1 + \frac{r^2}{a_n^2} + \dots + \frac{r^{2n-2}}{(n-1)!} \exp(-r^2/a_n^2) \quad (A-3)$$

where

$$a_n^2 = a_1^2 / n$$

For any of these functions it is necessary to determine the value of the parameter from a measured or estimated value of the average radius. Three values which have been used are:

- (a) the average  $\bar{r}$
- (b) the median  $R_m$
- (c) the equivalent radius  $R_e$  (defined below)

The average is given by

$$\bar{r} = \int_0^\infty r p(r) dr = \int_0^\infty r p(r) dr$$

For the Gaussian function  $p_1(r)$  the average is

$$\bar{r} = \frac{a_1}{\sqrt{\pi}} \sqrt{\pi} = 0.89 a_1$$

For  $p_2(r)$  the average is

$$\bar{r} = \frac{3a_2}{4} \sqrt{\frac{\pi}{2}} = 0.94 a_2$$

and for  $p_n(r)$  it is

$$\bar{r} = \frac{1 \cdot 3 \cdot 5 \dots (2n-1) a_n}{2^n (n-1)!} \sqrt{\frac{\pi}{2}}$$

As  $\alpha$  increases,  $\bar{X}_\alpha$  approaches  $\bar{X}$

The median  $\bar{X}_m$  is defined by

$$p(\bar{X}_m) = 1/2$$

For the Gaussian function the median is

$$\bar{X}_m = \bar{X} - \frac{1}{2} \log 2 = 1.85 \bar{X}$$

For  $p_1(x)$  it is  $1.85 \bar{X}_1$  approximately. For  $p_2(x)$  the parameter  $\bar{X}_2$  is greater than the median, which it approaches in the limit.

The equivalent damage radius  $\bar{X}_e$  is defined by the equation

$$1 = \int_0^\infty p(x) dx = \int_0^\infty \frac{1}{\bar{X}_e} dx$$

The quantity on the left is the expected number of targets damaged if they are uniformly distributed with unit density over a large region about the burst point. The quantity on the right is the corresponding number for a definite-range function of radius  $\bar{X}_e$ . The parameters  $\bar{X}_1$  have been introduced above in the form for which the parameter is always equal to  $\bar{X}_e$ , that is,  $\bar{X}_1 = \bar{X}_e$  for all  $\alpha$ .

To compare these functions we assume that the average  $\bar{X}$  is known and the parameters have been determined in terms of  $\bar{X}$ . The density functions  $p_1(\bar{X})$ ,  $p_2(\bar{X})$  and  $p_3(\bar{X})$  are plotted in figure 2-1. The corresponding probability functions and the definite-range function are plotted in figure 2-2. As  $\alpha \rightarrow \infty$ ,  $p_1(x)$  approaches the definite-range function. A measure of how closely  $\bar{X}$  is distributed about the mean is the probability that  $\bar{X}$  lies between  $0.5\bar{X}$  and  $1.5\bar{X}$ . This value is 75 percent for  $p_1(x)$ , 83 percent for  $p_2(x)$ , and 91 percent for  $p_3(x)$ .

Another measure of dispersion is the standard deviation from the mean. We have

$$\int_0^\infty \bar{X}^2 p(\bar{X}) d\bar{X} = \int_0^\infty \bar{X}^2 dp(\bar{X})$$

Integrating by parts, and noting that, for all the functions considered,  $p(\bar{X}) \rightarrow 0$  faster than  $\bar{X}^{-2}$  as  $\bar{X} \rightarrow \infty$ , this becomes

$$1 \int_0^\infty \bar{X} p(\bar{X}) d\bar{X} = \bar{X}_e^2 \quad (= \bar{X}_1^2)$$

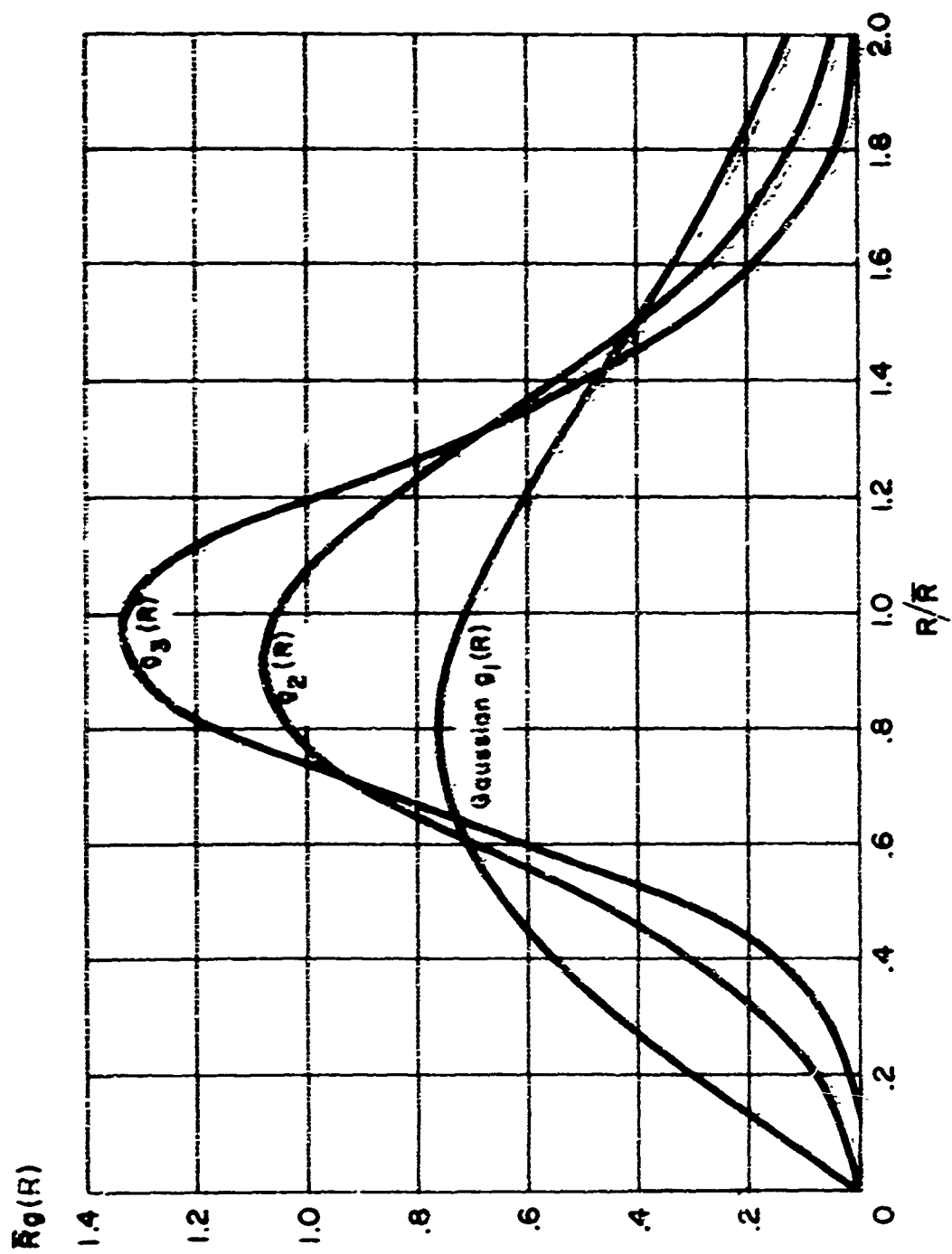


FIG. A-1: DENSITY FUNCTIONS OF DAMAGE RADIUS  $R$



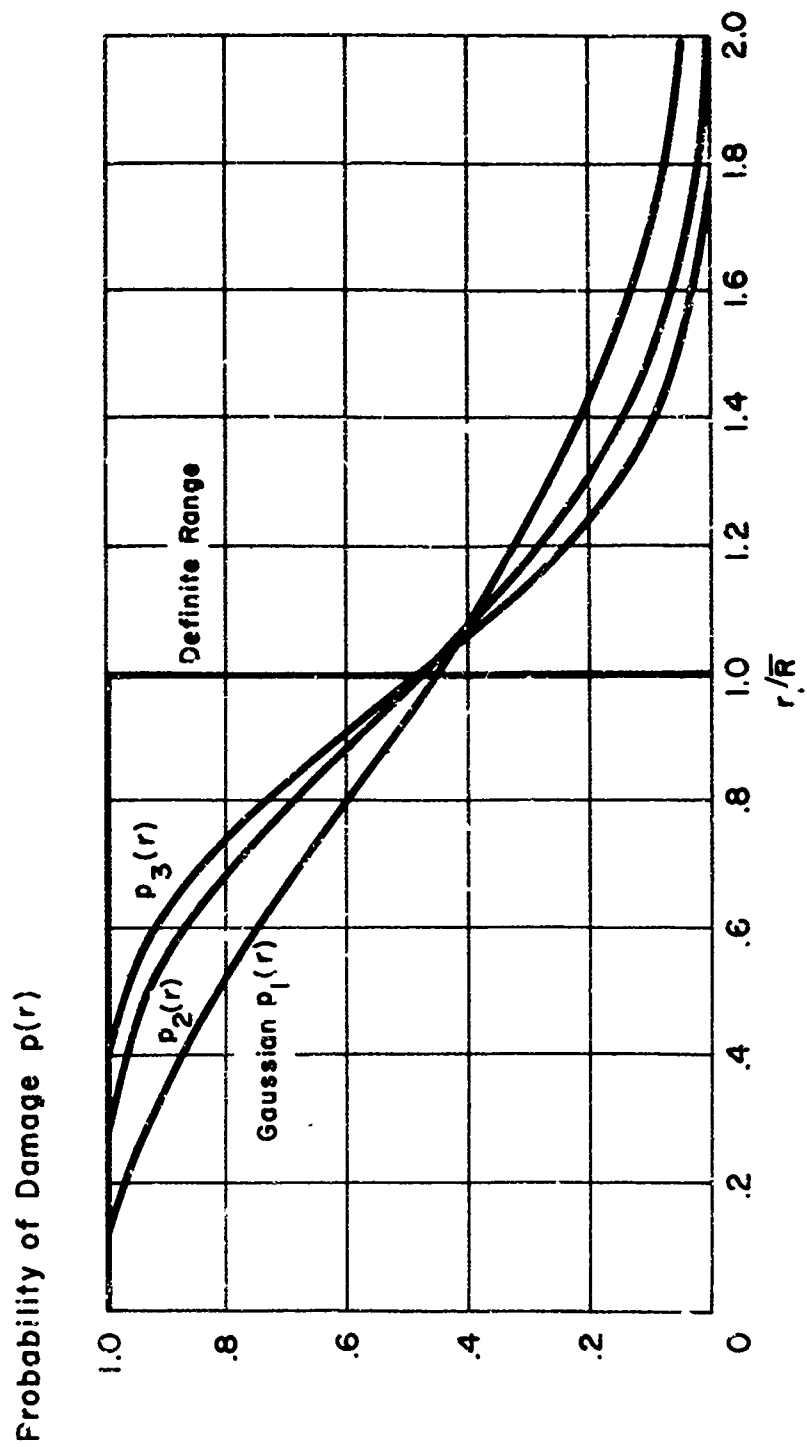


FIG. A-2: PROBABILITY OF DAMAGE AS FUNCTION OF HORIZONTAL RANGE  $r$

Hence, the variance from the mean is  $R_e^2 - \bar{R}^2$ . The standard deviation from the mean is  $0.52\bar{R}$ ,  $0.36\bar{R}$ , and  $0.29\bar{R}$  for  $p_1(r)$ ,  $p_2(r)$ , and  $p_3(r)$  respectively.

If we know  $\bar{R}$  and the standard deviation of  $R$  from  $\bar{R}$ , the appropriate function can be chosen from the sequence of functions. If only  $\bar{R}$  is known, how sensitive is the probability of damage to the choice of conditional damage function?

Let  $(x, y)$  be the rectangular coordinates of ground zero relative to the center of the target. Assume that  $x$  and  $y$  are independent and normally distributed with the same standard deviation  $\sigma$  and means  $h_x$  and  $h_y$  respectively. (Unequal standard deviations make the formulas more complicated but do not increase the difficulties of integration.) The density function of  $x$  and  $y$  is

$$f(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2} \left[ (x-h_x)^2 + (y-h_y)^2 \right]}$$

Then the average probability of damage for the  $n^{\text{th}}$  function is

$$P_n = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(x, y) f(x, y) dx dy$$

where  $p_n(x, y)$  is the function given in equation (A-3) when  $r^2$  is replaced by  $(x^2 + y^2)$ .

Expanding the powers of  $(x^2 + y^2)$  and completing the squares in the exponent, it is evident that the integral in  $P_n$  can be computed for any  $n$ . We omit details and give the result after the integration is completed. At this stage we have

$$P_n = e^{-H_n} (1 - \alpha_n) S_n \quad (\text{A-4})$$

where

$$S_n = \sum_{k=0}^{n-1} \sum_{j=k}^{n-1} \frac{\alpha_n^j}{2^j \cdot j!} S_{jk} \beta_n^k \quad (\text{A-5})$$

$$\frac{1}{\alpha_n} = 1 + \frac{\alpha_n^2}{2n\sigma^2} = 1 + \frac{2^{2n-1} [(n-1)!]^2}{\pi [1 \cdot 3 \dots (2n-1)]^2} \left( \frac{\bar{R}}{\sigma} \right)^2 \quad (\text{A-6})$$

$$H_n = \frac{k^2}{2\sigma^2} a_n \quad (A-7)$$

$$\beta_n = \left( \frac{i}{\sigma_n} - 1 \right) H_n$$

and  $S_{jk}$  is a sum which does not depend on  $n$ . The quantity  $k$  is the horizontal separation between intended ground zero and the target.

The factor  $S_{jk}$  reduces to

$$S_{jk} = \frac{i}{k!} \binom{j}{k} 2^j$$

Putting this into (A-5) we have that

$$\begin{aligned} S_n &= \sum_{k=0}^{n-1} \sum_{j=k}^{n-1} \binom{j}{k} a_n^j \frac{\beta_n^k}{k!} \\ &= \sum_{k=0}^{n-1} \frac{(\sigma_n \beta_n)^k}{(k!)^2} U_n^{(k)}(a_n) \end{aligned}$$

where  $U_n^{(k)}(a_n)$  is the  $k$ -th derivative of

$$U_n(a_n) = i + a_n + a_n^2 + \dots + a_n^{n-1}$$

Finally, the expression for  $P_n$  becomes

$$P_n = e^{-H_n} (1 - a_n) \sum_{k=0}^{n-1} \frac{(i - a_n)^k}{(k!)^2} U_n^{(k)}(a_n) H_n^k$$

with  $a_n$  and  $H_n$  as in (A-6) (A-7). The first few of these are

$$P_1 = e^{-H_1} (1 - a_1)$$

$$P_2 = e^{-H_2} (1 - a_2) \left[ (1 + a_2) + (1 - a_2) H_2 \right]$$

The extreme members of this sequence are obtained with  $z = 1$  (Gaussian) and  $z = \infty$  (definite-range). The values of  $P_1$ ,  $P_2$ , and  $P_\infty$  are listed for some values of  $k/\sigma$  and  $k/\sigma_\infty$  in table A-1. For small values of  $k/\sigma$ , the value of  $P_1$  is greater than that of  $P_\infty$ , while the converse is true for large values of  $k/\sigma$ . The value of  $k/\sigma$  at which they are equal is 1.0 for  $k/\sigma_\infty = 2$ , 1.1 for  $k/\sigma_\infty = 1$ , and about 2.0 for  $k/\sigma_\infty = 2$  or more.

The difference between  $P_1$  and  $P_\infty$  is as high as 21 percent in extreme cases. Hence, if we are interested in accuracies better than this, it is important to determine the approximate shape of the constructed damage curve.

However, the value of  $P_2$  seldom differs from either extreme by more than 13 percent. Therefore,  $P_2$  is a good value to use in those cases in which little is known about the shape of the curve except that it is not as gradual as the Gaussian curve or as abrupt as the definite-range (rectangular) curve.

TABLE A-1  
COMPARISON OF PROBABILITIES  
(percent)

$k/\sigma$	1			2			3		
	$P_1$	$P_2$	$P_\infty$	$P_1$	$P_2$	$P_\infty$	$P_1$	$P_2$	$P_\infty$
0.5	14	15	12	39	38	37	93	92	92
1.0	35	39	39	29	28	27	11	10	08
1.5	59	63	65	48	45	51	26	24	21
2	72	78	87	62	67	73	41	41	40
3	85	92	95	79	86	95	62	69	79
4	91	97	100	87	94	100	76	85	97

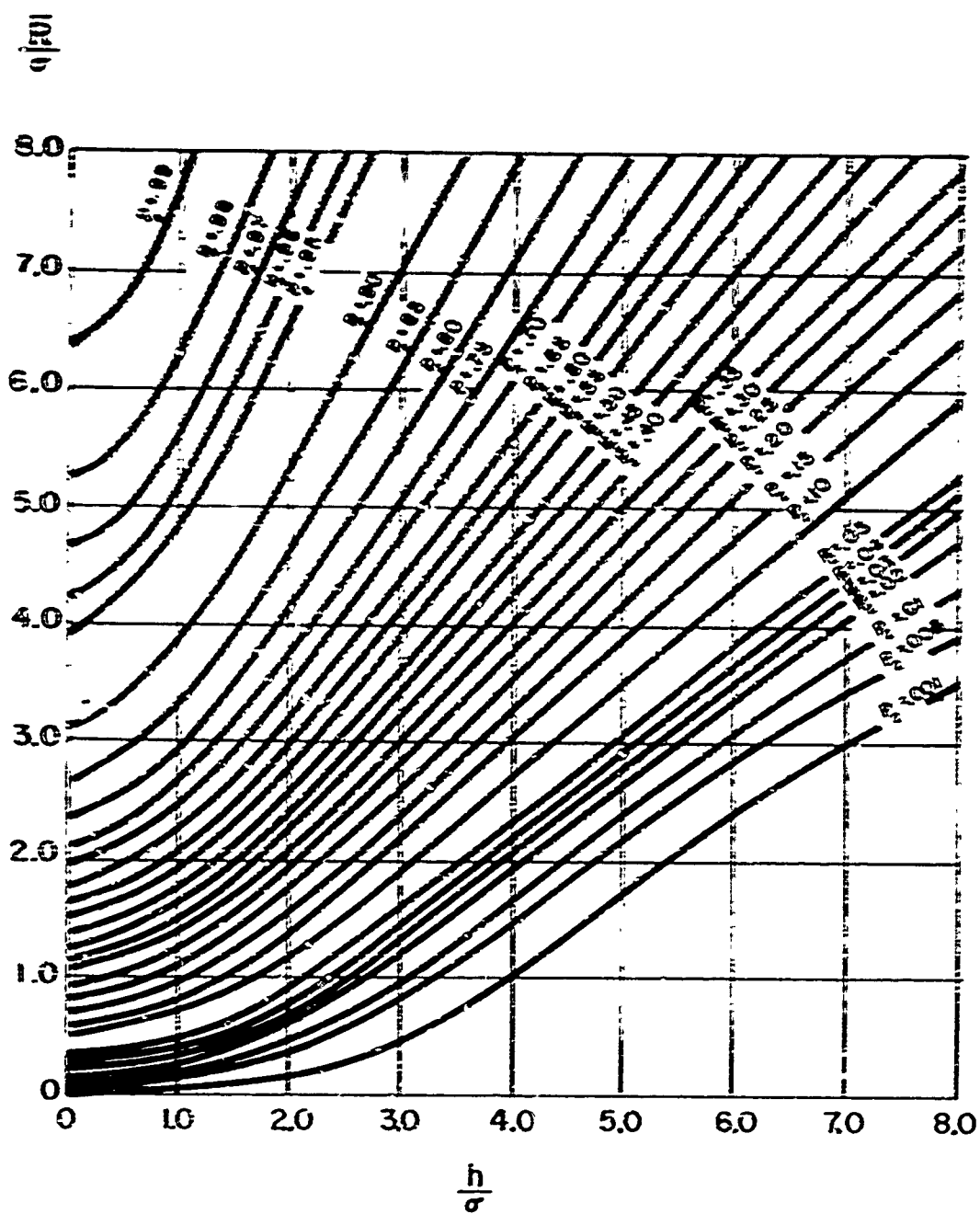


FIG. A-3: CONSTANT PROBABILITY CURVES FOR  $P_2$